Performance evaluation of demodulation with diversity — A combinatorial approach III: Threshold analysis

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Abstract

Log-likelihood of received bits is considered for the classical approach to demodulation of digital signals modulated using Binary Phase Shift Keying (BPSK). Using the symmetric functions mechanism a stable and efficient algorithm is obtained allowing to compute the probability that this value is close to zero. A combinatorial interpretation of the formulae obtained with symmetric functions is given by introducing a new class of combinatorial objects, and finally we give a bijection between this class and $\{0,1\}$-matrices.

1 Introduction

Modulating numerical signals means transforming them into wave forms. Due to their importance in practice, modulation methods were widely studied in signal processing (see for instance all the chapter 5 of [14]). One of the most important problems in this area is the performance evaluation of the optimum receivers associated with a given modulation method, which leads to the computation of various probabilities of errors (see again [14]).

Among the different modulation protocols used in practical contexts, an important class consists in methods where the modulation reference (i.e. a fixed numerical sequence) is transmitted over the same channel as the data signal. The demodulation decision is based then on at least two noisy informations, i.e. the transmitted signal and the transmitted reference. It happens however that one can also take into account in the demodulating process several noisy copies of these last two signals: one speaks then of demodulation with diversity. It appears that the probability of errors appearing in such contexts is of the following form:

$$P(U < V) = P\left(U = \sum_{i=1}^{N} |u_i|^2 \ < \ V = \sum_{i=1}^{N} |v_i|^2 \right), \tag{1.1}$$

where the u_i and v_i's denote independent centered complex Gaussian random variables with variances equal to $E[\,|u_i|^2\,] = \chi_i$ and $E[\,|v_i|^2\,] = \delta_i$ for every $i \in [1, N]$ (cf also Section 2).

The problem of computing explicitely probabilities of this last type was studied in signal processing by several researchers (cf [1, 8, 14, 16]). The most interesting result in this direction is due to Barett (cf [1]) who obtained the following expression

$$P(U < V) = \sum_{k=1}^{N} \left(\prod_{i \neq k} \frac{1}{1 - \delta_k^{-1}\delta_i} \prod_{i=1}^{N} \frac{1}{1 + \delta_k^{-1}\chi_i} \right),$$

for the probability given by formula (1.1).

In this article we consider the *log-likelihood* of a bit. This value allows to decide what was the value of the transmitted bit, and is also essential for various decoding algorithms such as MAP and its variants, and SOVA (see Chapter 4 of [7]).

In Section 2 we present the model describing the Binary Phase Shift Keying (BPSK) modulation that we use for our studies. Using the symmetric functions mechanism we obtain in Section 3 a stable and efficient algorithm that allows to compute the probability of the log-likelihood being close to zero, then in Section 4 we give a combinatorial interpretation of the obtained formulae and introduce a new class of combinatorial objects. Finally, we give a bijection between this class and a subclass of $\{0,1\}$-matrices. We conclude by proving a theorem caracterising the above subclass.

2 Signal processing background

We consider a model where one transmits an information $b \in \{-1, +1\}$ on a noisy channel [1]. A reference $r = 1$ is also sent on the noisy channel at the same time as b. We assume that we receive N pairs $(x_i(b), r_i)_{1 \le i \le N} \in (\mathbb{C} \times \mathbb{C})^N$ of data (the $x_i(b)$'s) and

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[1] This situation corresponds to Binary Phase Shift Keying (BPSK).

references (the r_i's) [2] that have the following form

$$\left\{ \begin{array}{rcll} x_i(b) & = & a_i\, b + \nu_i & \text{for every } 1 \le i \le N, \\ r_i & = & a_i \sqrt{\beta_i} + \nu_i' & \text{for every } 1 \le i \le N, \end{array} \right.$$

where i is the path number, $a_i \in \mathbb{C}$ is a complex number that models the channel fading [3] associated with the i-th path, $\beta_i \in \mathbb{R}^+$ is a positive real number that represents the signal to noise ratio (SNR) available for the reference r_i, and $\nu_i \in \mathbb{C}$ and $\nu_i' \in \mathbb{C}$ denote two independent complex white Gaussian noises. We also assume that every a_i is a complex random variable distributed according to a centered Gaussian density of variance α_i for every $i \in [1, N]$.

According to these assumptions, all observables of our model, i.e. the pairs $(x_i(b), r_i)_{1\le i\le N}$, are complex Gaussian random variables. We finally also assume that these N observables are N independent random variables which have their image in $\mathbb{C}^2$. Under these hypotheses we have the following formula for the *log-likelihood* that serves as a decision variable in BPSK.

$$\Lambda = \log \frac{P(b=+1|X)}{P(b=-1|X)} = \sum_{i=1}^{N} \frac{4\,\alpha_i \sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)}\,(x_i(b)|r_i) \tag{2.2}$$

with $X = (x_i(b), r_i)_{1\le i\le N}$ and where $(\star|\star)$ denotes the Hermitian scalar product. One indeed decides that b was equal to 1 (resp. to -1) when $\Lambda > 0$ (resp. $\Lambda < 0$), while the actual value of log-likelihood is used in Turbo decoding algorithms. One obtains (1.1) by applying the parallelogram identity to (2.2).

The situation undesirable for both demodulation (increased chances of taking incorrect decision) and soft decoding algorithms (unreliable input) is when the log-likelihood is close to zero, i.e. $|\Lambda| < \varepsilon$. We shall therefore study the probability $P(U - V < \varepsilon)$ [4] generalising (1.1) where $P(U - V < 0)$ is considered instead. Here, U and V are two real random variables defined as above. It is then easy to prove in the same manner as in [3] the following analogue of Barret's formula:

$$P(U-V<\varepsilon) = \sum_{k=1}^{N} \frac{\delta_k^{2N-1} e^{\frac{\varepsilon}{\delta_k}}}{\prod\limits_{1\le i\le N}\Big(\delta_k+\chi_i\Big)\prod\limits_{1\le i\ne k\le N}\Big(\delta_k-\delta_i\Big)} \tag{2.3}$$

3 Symmetric functions expression

We refer the reader to [3] for all details related to symmetric functions that we use in this section. The right hand side of (2.3) being symmetric in δ_i and χ_i, we shall represent it in terms of Schur functions. Replacing the exponential in the numerator by its Taylor decomposition, we obtain

$$P(U-V<\varepsilon) = \sum_{m=0}^{+\infty}\sum_{k=1}^{N} \frac{\delta_k^{2N-m-1}}{\prod\limits_{1\le i\le N}\Big(\delta_k+\chi_i\Big)\prod\limits_{1\le i\ne k\le N}\Big(\delta_k-\delta_i\Big)} \times \frac{\varepsilon^m}{m!}\,. \tag{3.4}$$

We will now concentrate our efforts on the m-th coefficient of this exponential series, i.e.

$$P_m^{(N)}(\Delta, X) = \sum_{k=1}^{N} \frac{\delta_k^{2N-m-1}}{\prod\limits_{1\le i\le N}\Big(\delta_k+\chi_i\Big)\prod\limits_{1\le i\ne k\le N}\Big(\delta_k-\delta_i\Big)}\,. \tag{3.5}$$

For simplicity we shall abbreviate $P_m^{(N)}(\Delta, X)$ to $P_m^{(N)}$. Keeping the same notation as in [3], we can express this formula using the Lagrange operator L. Let us indeed set $\delta_k = x_k$ and $\chi_k = -y_k$ for every $k \in [1, N]$. Then one can rewrite (3.5) as

$$P_m^{(N)} = \sum_{k=1}^{N} \frac{x_k^{2N-m-1}}{R(x_k, Y)R(x_k, X\backslash x_k)}$$

where we denoted $X = \{\, x_1, \ldots, x_N \,\}$ and $Y = \{\, y_1, \ldots, y_N \,\}$ and where

$$R(A, B) = \prod_{a\,\in\,A, b\,\in\,B} (a - b)$$

is the resultant of two polynomials having A and B as sets of roots. Hence we have

$$P_m^{(N)} = \sum_{k=1}^{N} \frac{g(x_k, X\backslash x_k)}{R(x_k, X\backslash x_k)} = L(g) \tag{3.6}$$

where g stands for the element of $Sym(x_1)\otimes Sym(X\backslash x_1)$ defined by setting

$$g(x_1, X\backslash x_1) = g(x_1) = \frac{x_1^{2N-m-1}}{R(x_1, Y)}\,.$$

[2] One speaks in this case of spatial diversity, i.e. when more than one antenna is available, but also of multipath reflexion contexts. These two types of situations typically occur in mobile communications.

[3] Fading is typically the result of the absorption of the signal by buildings. Its complex nature comes from the fact that it models both an attenuation (its modulus) and a dephasing (its argument).

[4] Probability $P(U - V < \varepsilon)$ can be studied independently as the distribution function of the random variable $U - V$ (cf. [6]).

Observe now that one has

$$g(x_1, X\backslash x_1) = \frac{1}{R(X,Y)}\, x_1^{2N-m-1} f(x_1, X\backslash x_1)$$

where f stands for the element of $Sym(x_1) \otimes Sym(X\backslash x_1)$ defined by setting

$$\begin{aligned} f(x_1, X\backslash x_1) &= R(X\backslash x_1, Y) \\ &= s_{(N^{N-1})}((X\backslash x_1) - Y) \end{aligned} \tag{3.7}$$

(the last above equality comes from the expression of the resultant in terms of Schur functions). Note now that the resultant $R(X,Y)$, being symmetric in the alphabet X, is a scalar for the operator L. It follows therefore from relation (3.6) that one has

$$P_m^{(N)} = \frac{L(x_1^{2N-m-1} f(x_1, X\backslash x_1))}{R(X,Y)}. \tag{3.8}$$

Let us now study the numerator of the right-hand side of relation (3.8) in order to give another expression for $P_m^{(N)}$. Note first that Cauchy formula leads to the development

$$s_{(N^{N-1})}((X\backslash x_1) - Y) = \sum_{\lambda \subset (N^{N-1})} s_\lambda(X\backslash x_1)\, s_{(N^{N-1})/\lambda}(-Y)\,. \tag{3.9}$$

According to the identities (3.7) and (3.9), we now obtain for $0 \le m < 2N$ the relations

$$\begin{aligned} &L(x_1^{2N-m-1} f(x_1, X\backslash x_1)) = \\ &= \sum_{\lambda \subset (N^{N-1})} L(x_1^{2N-m-1} s_\lambda(X\backslash x_1))\, s_{(N^{N-1})/\lambda}(-Y) \\ &= \sum_{\lambda \subset (N^{N-1})} s_{(\lambda, N-m)}(X)\, s_{(N^{N-1})/\lambda}(-Y), \end{aligned}$$

the last above equality being an immediate consequence of the following theorem:

THEOREM 3.1. (Lascoux; [11]) *Let $X = \{ x_1, \dots, x_N \}$ be an alphabet consisting of N indeterminates and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition that contains $\rho_{N-1} = (0, 1, 2, \dots, N-2)$. Then one has*

$$L(x_1^k\, s_\lambda(X\backslash x_1)) = s_{\lambda, k-N+1}(X) \tag{3.10}$$

for every $k \ge 0$, where the Schur function involved in the right hand side of relation (3.10) is indexed by the sequence $(\lambda, k-N+1) = (\lambda_1, \dots, \lambda_n, k-N+1)$ of $\mathbb{Z}^{n+1}$.

Using the equality

$$s_{\lambda/\mu}(-X) = s_{\tilde{\lambda}/\tilde{\mu}}(X),$$

where $\mu \subset \lambda$ (see [13]) and going back to the definition of skew Schur functions, we can rewrite the last above expression as

$$\begin{aligned} &L(x_1^{2N-m-1} f(x_1, X\backslash x_1)) = \\ &= \sum_{\lambda \subset (N^{N-1})} (-1)^{|\lambda|} s_{(\lambda, N-m)}(X)\, s_{\overline{(\lambda, N)}}(Y), \end{aligned}$$

where $0 \le m < 2N$, and $\overline{(\lambda, N)}$ denotes the complementary partition of (λ, N) in the square (N^N). Going back to the initial variables, the signs disappear in the previous formula by homogeneity of Schur functions. Reporting the identity obtained in such a way into relation (3.8), we finally get an expression for $P_m^{(N)}$ in terms of Schur functions, i.e.

$$P_m^{(N)} = \frac{\displaystyle\sum_{\lambda \subset (N^{N-1})} s_{(\lambda, N-m)}(\Delta)\, s_{\overline{(\lambda, N)}}(X)}{\displaystyle\prod_{1 \le i,j \le N} (\chi_i + \delta_j)} \tag{3.11}$$

where $X = \{\chi_1, \dots, \chi_N\}$, $\Delta = \{\delta_1, \dots, \delta_N\}$. A slightly different expression can be obtained for $m \ge 2N$, but the first $2N$ coefficients of decomposition (3.4) already provide sufficient precision. In this paper we shall therefore limit our study to $0 \le m < 2N$.

3.1 A Toeplitz system and its solution Going back to the alphabets X and Y defined in Section 3, one can obtain the following expression for $P_m^{(N)}$ in terms of multi Schur functions:

$$P_m^{(N)} = \frac{s_{(N^{N-1}, N-m)}(X-Y, \dots, X-Y, X)}{s_{(N^N)}(X-Y, \dots, X-Y)} \tag{3.12}$$

where the alphabet $X - Y$ appears $N-1$ times in the numerator and N times in the denominator of the right hand side of the above formula.

Using the determinantal expression for multi Schur functions, we can now observe that relation (3.12) shows that $P_m^{(N)}$ is equal to the quotient of the determinant

$$\begin{vmatrix} s_N(Z) & s_{N+1}(Z) & \dots & s_{2N-2}(Z) & s_{2N-m-1}(X) \\ s_{N-1}(Z) & s_N(Z) & \dots & s_{2N-3}(Z) & s_{2N-m-2}(X) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_2(Z) & s_3(Z) & \dots & s_N(Z) & s_{N-m+1}(X) \\ s_1(Z) & s_2(Z) & \dots & s_{N-1}(Z) & s_{N-m}(X) \end{vmatrix},$$

where Z is a shorthand notation for $X - Y$, by the determinant

$$\begin{vmatrix} s_N(Z) & s_{N+1}(Z) & \dots & s_{2N-1}(Z) \\ s_{N-1}(Z) & s_N(Z) & \dots & s_{2N-2}(Z) \\ \vdots & \vdots & \ddots & \vdots \\ s_1(Z) & s_2(Z) & \dots & s_N(Z) \end{vmatrix}$$

obtained by replacing the last column of the first determinant by the N-dimensional vector $(s_{2N-1}(Z), s_{2N-2}(Z), \ldots, s_N(Z))$. Hence the right-hand side of relation (3.12) can be interpreted as the Cramer expression for the last component p_0 of the linear system

$$\begin{pmatrix} s_N(Z) & s_{N+1}(Z) & \ldots & s_{2N-1}(Z) \\ s_{N-1}(Z) & s_N(Z) & \ldots & s_{2N-2}(Z) \\ \vdots & \vdots & \ddots & \vdots \\ s_1(Z) & s_2(Z) & \ldots & s_N(Z) \end{pmatrix} \begin{pmatrix} p_{N-1} \\ p_{N-2} \\ \vdots \\ p_0 \end{pmatrix} = \\ = \begin{pmatrix} s_{2N-m-1}(X) \\ s_{2N-m-2}(X) \\ \vdots \\ s_{N-m}(X) \end{pmatrix} .$$

Let us now consider the generating series of complete and elementary symmetric functions, defined by setting

$$\begin{aligned} \sigma_t(X) &= \sum_{n=0}^{+\infty} S_n(X)\, t^n = \prod_{x\in X} \frac{1}{1-x\,t} \\ \lambda_t(X) &= \sum_{n=0}^{+\infty} \Lambda_n(X)\, t^n = \prod_{x\in X} (1+x\,t)\ . \end{aligned}$$

Taking also $\pi_m(t) = p_0 + p_1\,t + \cdots + p_{N-1}\,t^{N-1}$, the above linear system implies that the coefficients of order N to $2N-1$ in the series $\pi_m(t)\,\sigma_t(X-Y)$ are equal to the coefficients of the same order in $t^m\sigma_t(X)$, where $\sigma_t(X-Y)$ is defined by

$$\sigma_t(X-Y) = \sum_{n=0}^{+\infty} S_n(X-Y)\, t^n = \sigma_t(X)\,\lambda_{-t}(Y)\ .$$

This means equivalently that there exists a polynomial $\mu_m(t)$ of degree less than or equal to $N-1$ such that one has

$$\pi_m(t)\,\sigma_t(X-Y) - t^m\sigma_t(X) + \mu_m(t) = O(t^{2N})\ .$$

Going back to the definition of $\sigma_t(X-Y)$, one can notice that this property can be rewritten as

$$(\pi_m(t)\lambda_{-t}(Y) - t^m)\sigma_t(X) + \mu_m(t) = O(t^{2N})$$

that is itself clearly equivalent to

$$\pi_m(t)\lambda_{-t}(Y) + \mu_m(t)\lambda_{-t}(X) = t^m + O(t^{2N}).$$

Since the left hand side of the above identity is a polynomial of degree at most $2N-1$, it follows that its right hand side must be equal to t^m, keeping in mind that we consider $0 \le m < 2N$. Hence we showed that one has

$$\pi_m(t)\lambda_{-t}(Y) + \mu_m(t)\lambda_{-t}(X) = t^m. \tag{3.13}$$

Thus, $P_m^{(N)}$ is the constant term $\pi_m(0)$ of $\pi_m(t)$, where $\pi_m(t)$ and $\mu_m(t)$ are the polynomials of degree $\le N-1$ defined by (3.13).

3.2 A Bezoutian algorithm We now present an algorithm that computes π_m and μ_m iteratively, starting with $m = 0$, and then consequetively deriving π_m and μ_m from π_{m-1} and μ_{m-1} for $m = 1, \ldots, 2N-1$.

Algorithm 3.1. (Calculating the polynomials π_m and μ_m)
Input: Two alphabets $\Delta = \{\delta_1, \ldots, \delta_N\}$ *and* $X = \{\chi_1, \ldots, \chi_N\}$.
Output: For all $m = 0, \ldots, 2N-1$, *a pair of polynomials* (π_m, μ_m) *satisfying (3.13).*

For $m = 0$, the right hand side of the equality (3.13) is 1, i.e. the greatest common divisor of $\lambda_{-t}(Y)$ and $\lambda_{-t}(X)$. This implies that we can use the Generalised Euclidean algorithm as first step of our algorithm.

- **Step 0.1.** Consider the two polynomials $X(t)$ and $\Delta(t)$ of $\mathbb{R}[t]$ defined by setting

$$X(t) = \prod_{i=1}^{N} (1-\chi_i\,t) \quad \text{and} \quad \Delta(t) = \prod_{i=1}^{N} (1+\delta_i\,t).$$

- **Step 0.2.** Compute the unique polynomial $\pi_0(t)$ of $\mathbb{R}[t]$ of degree $d(\pi_0) \le N-1$ such that

$$\pi_0(t)X(t) + \mu_0(t)\Delta(t) = 1$$

  where $\mu_0(t)$ stands for some polynomial of $\mathbb{R}[t]$ of degree $d(\mu_0) \le N-1$.

Suppose now that at `Step` $m-1$ we have found the polynomials $\pi_k(t)$ and $\mu_k(t)$ for all $k < m$. Then the following `Step` m provides us the next pair of polynomials $\pi_m(t)$ and $\mu_m(t)$ of degrees $\le N-1$, satisfying the relation (3.13).

- **Step m.1.** We suppose that $0 < m < 2N$. Let then

$$\begin{aligned} c &= \frac{[t^{N-1}](\mu_{m-1})}{\chi_1 \cdots \chi_N} \\ &= (-1)^{N-1} \frac{[t^{N-1}](\pi_{m-1})}{\delta_1 \ldots \delta_N}\ , \end{aligned} \tag{3.14}$$

  where $[t^{N-1}](\pi)$ stands for the coefficient of t^{N-1} in the polynomial $\pi(t)$.

- **Step m.2.** To obtain the required polynomials we then define

$$(3.15)\quad \begin{cases} \pi_m(t) = t\,\pi_{m-1}(t) + (-1)^N c\,\Delta(t) \\ \mu_m(t) = t\,\mu_{m-1}(t) - (-1)^N c\,X(t) \end{cases}$$

PROPOSITION 3.1. *The polynomials $\pi_m(t)$ and $\mu_m(t)$, produced by Algorithm 3.1, satisfy relation (3.13).*

Proof – We argue by induction on m. The proposition being obvious for $m = 0$, provides the base case. Observe that it follows immediately from (3.13) and the fact that $m - 1 < 2N - 1$, that if $d(\pi_{m-1}) = N - 1$ then also $d(\mu_{m-1}) = N - 1$ and vice versa. If we have $d(\pi_{m-1}), d(\mu_{m-1}) < N - 1$, then the two polynomials $\pi_m(t) = t\,\pi_{m-1}(t)$ and $\mu_m(t) = t\,\mu_{m-1}(t)$ satisfy (3.13) for m. Thus the inductive step is essentially reduced to verifying that in the case where both $d(\pi_{m-1}) = N - 1$ and $d(\mu_{m-1}) = N - 1$, the polynomials constructed in `Step` m.2 have the necessary properties. This verification is rather straightforward so we shall not present it here. ∎

NOTE 3.1. We recall that $\pi_m(0) = P^{(N)}_m$.

4 Bijective approach

4.1 A special case Recall that originally $P^{(N)}_m$ has been defined as the m-th coefficient of the decomposition of $P(U - V < \varepsilon)$ into an exponential series (cf. Section 3):

$$P(U - V < \varepsilon) = \sum_{m=0}^{\infty} P^{(N)}_m \times \frac{\varepsilon^m}{m!}\,,$$

and therefore, taking $\varepsilon = 0$, we obtain (see also (3.11))

$$P(U < V) = P^{(N)}_0 = \frac{\sum_{\lambda\subset(N^{N-1})} s_{(\lambda,N)}(\Delta)\, s_{\overline{(\lambda,N)}}(X)}{\prod_{1\le i,j\le N}\left(\chi_i + \delta_j\right)}\,. \tag{4.16}$$

This expression for the probability $P(U < V)$ has been obtained in [3], while in [10] it has been given a combinatorial interpretation. Indeed, as $\lambda \subset (N^{N-1})$ we have $\lambda_i \le N$ for all i, and thus (λ, N) is also a partition.

It is well known that a Schur function over an alphabet A indexed by some partition λ can be expressed as a sum

$$s_\lambda(A) = \sum_{t_\lambda} m(t_\lambda)\,,$$

where t_λ runs through all possible Young tableaux over A of shape λ, and $m(t_\lambda)$ is the monom obtained by taking the product of all elements of A contained in t_λ. For example,

$$s_{(1,2)}(\{\chi_1, \chi_2\}) = \chi_1^2\chi_2 + \chi_1\chi_2^2$$

corresponds to

$$\begin{array}{|c|c} \hline \chi_2 & \multicolumn{1}{c}{} \\ \hline \chi_1 & \multicolumn{1}{c|}{\chi_1} \\ \hline \end{array} + \begin{array}{|c|c} \hline \chi_2 & \multicolumn{1}{c}{} \\ \hline \chi_1 & \multicolumn{1}{c|}{\chi_2} \\ \hline \end{array}\,.$$

In this manner one can represent the numerator of the fraction in (4.16) as a sum of the monoms corresponding to (N^N) square tabloids consisting of a Young tableau over the alphabet Δ and a complimentary one over X. For example,

$$P^{(2)}_0 = \frac{\chi_1\chi_2(\delta_1^2 + \delta_1\delta_2 + \delta_2^2) + (\chi_1 + \chi_2)(\delta_1^2\delta_2 + \delta_1\delta_2^2) + \delta_1^2\delta_2^2}{(\chi_1 + \delta_1)(\chi_1 + \delta_2)(\chi_2 + \delta_1)(\chi_2 + \delta_2)}$$

can be represented as the following sum

$$\begin{array}{|c|c|}\hline \chi_2 & \chi_1 \\ \hline \delta_1 & \delta_1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline \chi_2 & \chi_1 \\ \hline \delta_1 & \delta_2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline \chi_2 & \chi_1 \\ \hline \delta_2 & \delta_2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline \delta_2 & \chi_1 \\ \hline \delta_1 & \delta_1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline \delta_2 & \chi_2 \\ \hline \delta_1 & \delta_1 \\ \hline\end{array} +$$

$$+ \begin{array}{|c|c|}\hline \delta_2 & \chi_1 \\ \hline \delta_1 & \delta_2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline \delta_2 & \chi_2 \\ \hline \delta_1 & \delta_2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline \delta_2 & \delta_2 \\ \hline \delta_1 & \delta_1 \\ \hline\end{array}\,.$$

For an arbitrary m such that $0 < m < 2N$ it is possible that $(\lambda, N - m)$ (see again (3.11)) is not a partition. In order to obtain an analogous representation of $P^{(N)}_m$ we will have to introduce a more complex combinatorial object — *square tabloid with ribbon.*

4.2 Square tabloids with ribbons

DEFINITION 4.1. *A* ribbon *in a Young diagram is a connected chain of boxes not containing a 2×2 square such that any box has at most two neighbours (see Figure 1). The number of boxes in a ribbon is its* length.

The examples *a–c* in the Figure 1 are correct ribbons, while *d–f* are not. Here we will only consider those that start in the lower right-hand corner of the square, and go to the North-West (examples *b*, *c*). For the sake of simplicity we will omit these two conditions when speaking of ribbons.

We shall denote by $\mathcal{R}^{(N)}_m$ the set of all such ribbons of length m in (N^N). In a way analoguous to the one used to represent Young diagrams as a partition of an integer, a ribbon is fully described by a sequence $(r_1, \ldots, r_k)$, where r_i is the number of boxes in its i-th row. For

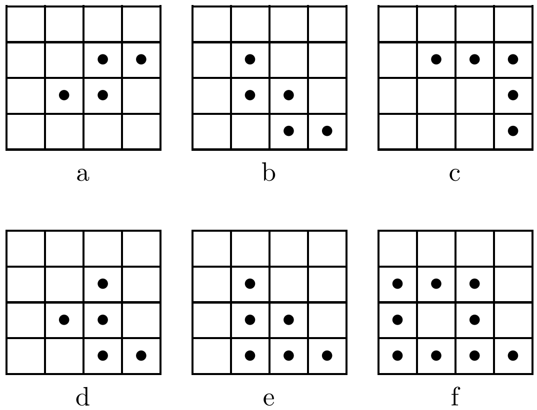


Figure 1: Several examples and counter-examples for the Definition 4.1 of ribbons.

example, the ribbons b and c in Figure 1 are represented by (2, 2, 1) and (1, 1, 3) correspondingly.

We call a *square tabloid with ribbon* a triplet consisting of a ribbon $r \in \mathcal{R}_m^{(N)}$ and two Young tableaux of shapes λ and μ on the alphabets $\Delta = \{\delta_1, \ldots, \delta_N\}$ and $X = \{\chi_1, \ldots, \chi_N\}$ correspondingly, such that put together they form a complete square (N^N). We denote by $\mathcal{T}_m^{(N)}$ the set of all such triplets:

$$\mathcal{T}_m^{(N)} = \{(t_\lambda, t_\mu, r) | r \in \mathcal{R}_m^{(N)}, \lambda \cup r \subset (N^N), \mu = \overline{\lambda \cup r}\},$$

where $1 \leq m \leq 2N-1$, while λ and μ are the shapes of t_λ, and t_μ (see Figure 2-a).

$$\begin{array}{|c|c|c|c|}\hline \delta_4 & \chi_4 & \chi_3 & \chi_1 \\ \hline \delta_3 & \bullet & \chi_3 & \chi_1 \\ \hline \delta_2 & \bullet & \bullet & \chi_4 \\ \hline \delta_1 & \delta_2 & \bullet & \bullet \\ \hline \end{array} \qquad \left(\begin{array}{cccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

a b

Figure 2: A typical element of $\mathcal{T}_4^{(5)}$ and the corresponding matrix.

Proposition 4.1. *We have the following representation for* $P_m^{(N)}$

$$P_m^{(N)} = \frac{\sum_{t \in \mathcal{T}_m^{(N)}} (-1)^{h(t)-1} m(t)}{\prod_{1 \leq i,j \leq N} \left(\chi_i + \delta_j\right)},$$

where $m(t)$ is the monom corresponding to t, and $h(t)$ is the height of the ribbon of t defined as follows. If $t = (t_\lambda, t_\mu, r) \in \mathcal{T}_m^{(N)}$ is a square tabloid with ribbon, and $r = (r_1, \ldots, r_N)$ then $h(t) = \max\{i | r_i > 0\}$.

Proof – This is an obvious consequence of (3.11), considering that $s_{(\ldots,i,j,\ldots)}(A) = -s_{(\ldots,j+1,i-1,\ldots)}(A)$. ∎

Example 4.1. The coefficient

$$P_2^{(2)} = \frac{\chi_1\chi_2 - \delta_1\delta_2}{(\chi_1 + \delta_1)(\chi_1 + \delta_2)(\chi_2 + \delta_1)(\chi_2 + \delta_2)}$$

corresponds to

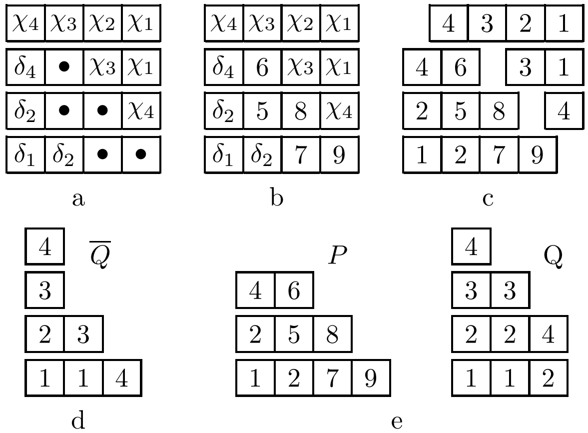
$$\begin{array}{|c|c|}\hline \chi_2 & \chi_1 \\ \hline \bullet & \bullet \\ \hline \end{array} - \begin{array}{|c|c|}\hline \delta_2 & \bullet \\ \hline \delta_1 & \bullet \\ \hline \end{array}.$$

4.3 Description of the bijection In the following we shall construct a bijection between the square tabloids with ribbons introduced in the previous section, and a subset $\mathfrak{M}_m^{(N)}$ of $\mathcal{M}_{N\times(N+m)}$[5] that we will define later on. We shall split a matrix from this set into two parts: the one on the left-hand side containing N columns, and the one on the right-hand side — m columns. The right part is the one that will eventually generate the ribbon in the corresponding tableau, and has only one 1 in each of its columns. Meanwhile the left part will be responsible for the tableau t_λ, and has one 1 for each of the boxes of t_λ. The matrix corresponding to the tabloid in Figure 2-a is shown in Figure 2-b.

We will now present an algorithm that, given a square tabloid $T \in \mathcal{T}_m^{(N)}$, will construct a matrix from $\mathcal{M}_{N\times(N+m)}$. As this algorithm is based on the Knuth correspondence (cf. [4, 9, 10]), it can be easily seen that it is reversible, thus, denoting by $\mathfrak{M}_m^{(N)}$ the image of the mapping defined by this algorithm, we obtain a bijection between $\mathfrak{M}_m^{(N)}$ and $\mathcal{T}_m^{(N)}$.

Algorithm 4.1. (Construction of a $\{0,1\}$-matrix corresponding to a square tabloid)
Input: $T = (t_\lambda, t_\mu, r)$ — a square tabloid in $\mathcal{T}_m^{(N)}$.
Output: A $\{0,1\}$-matrix $M \in \mathcal{M}_{N\times(N+m)}$.

- **Step 1.** Number each box of r starting with $N+1$ and up to $N+m$ from bottom to top and from left to right (see Figure 3-b).
- **Step 2.** Replace all δ_i in t_λ and χ_i in t_μ by i, and join t_λ with r to obtain two Young tableaux P and $\overline{Q}$ of shapes $\lambda \cup r$ and $\mu = \overline{\lambda \cup r}$ correspondingly (Figure 3-c).
- **Step 3.** Apply the same procedure as in [10] to obtain a tableau Q of a shape conjugated to $\lambda \cup r$ (Figure 3-d, e; see [10] for a formal description).
- **Step 4.** Apply Knuth's bijection based on column bumping to the pair (P, Q) of Young tableaux of conjugate forms to obtain a matrix from $\mathcal{M}_{N\times(N+m)}$ (Figure 3-f).



$$\left(\begin{array}{cccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

f

Figure 3: Applying the algorithm to an element of $\mathcal{T}_4^{(5)}$.

Example 4.2. Let us elaborate on the example given along the algorithm. We start with the square tabloid from $\mathcal{T}_4^{(5)}$ shown in Figure 3-*a*.

As the size of the square is 4, and the length of the ribbon is 5, the classical numbering for the ribbon goes from 5 to 9 and is shown in Figure 3-*b*. Re-labelling the rest of the tabloid and re-arranging it as indicated in the Step 2 of Algorithm 4.1 we obtain the two Young tableaux shown in Figure 3-*c*.

In Step 3 we take the upper right-hand corner tableau $\overline{Q}$ shown in Figure 3-*d*, and we transform it into another one of complementary shape by applying the complementation procedure described in [10] to obtain the tableau Q from Figure 3-*e*. In short, the column i of Q contains exactly those numbers from 1 to N that are not in the column $N-i$ of $\overline{Q}$. In this example $N=4$, and the last column of Q is empty, as the first one of $\overline{Q}$ contains all numbers between 1 and 4. The only thing left to do now is to apply Knuth's bijection to the pair (P, Q) to obtain the matrix in Figure 3-f. ∎

Note 4.1. We denote by $\mathfrak{M}^{(N)} = \bigcup_{m=1}^{2N-1} \mathfrak{M}_m^{(N)}$ the set of all matrices that can be obtained by applying this algorithm.

As it has been mentioned in the beginning of the section, it can be easily seen that Algorithm 4.1 is reversible. More precisely, we have the following reciprocal algorithm.

Algorithm 4.2. (Construction of a square tabloid corresponding to a $\{0,1\}$-matrix)
Input: A $\{0,1\}$-matrix $M \in \mathfrak{M}_m^{(N)}$.
Output: $T = (t_\lambda, t_\mu, r)$ — a square tabloid in $\mathcal{T}_m^{(N)}$.

- **Step 1.** Applying Knuth's bijection in the opposite direction we can transform any matrix from $\mathcal{M}_{N\times(N+m)}$ into a pair of Young tableaux P and Q of conjugated shapes: on the alphabets $\{1,\ldots,N+m\}$ and $\{1,\ldots,N\}$ correspondingly.

- **Step 2.** The fact that M belongs to $\mathfrak{M}_m^{(N)}$ implies that both P and Q fit into (N^N), and thus we can again apply to Q the same procedure as in [10] to obtain a new Young tableau t_μ of the shape complementary to that of P.

- **Step 3.** Once again referring to the fact that M belongs to $\mathfrak{M}_m^{(N)}$, we can state that in P there is exactly one occurence of each one of $N+1,\ldots,N+m$, and that the corresponding boxes form a ribbon r numbered from bottom to top and from left to right. Moreover, this ribbon can be cut out of P leaving a Young tableau t_λ on the alphabet $\{1,\ldots,N\}$. (See Section 4.4 for conditions on $\{0,1\}$-matrices defining $\mathfrak{M}_m^{(N)}$ explicitely.)

- **Step 4.** To finalise our algorithm it is sufficient to replace all entries i in t_λ with δ_i, and in t_μ — with χ_i.

Example 4.3. To reverse Example 4.2 we start with the matrix shown in Figure 3-f, and transform it into a two-row array representing the positions of 1's: in each column of the array the element in the first row is the row number, and the one in the second row — the column number of a position containing 1. We obtain therefore the following array:

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 7 & 9 & 2 & 5 & 8 & 1 & 6 & 2 & 4 \end{pmatrix}$$

Applying Knuth's bijection consists now in forming one Young tableau by column-bumping in the elements of the second row of the array from left to right, and placing the corresponding elements of the first row into a Young tableau of the conjugated form. This results exactly in the pair of tableaux shown in Figure 3-*e*. The rest of the algorithm consists in retracing backwards the first steps of Algorithm 4.1 to obtain the tabloid in Figure 3-*a*. ∎

[5] In this paper we only consider $\{0,1\}$-matrices, therefore we use $\mathcal{M}_{m\times n}$ as a shorthand notation for $\mathcal{M}_{m\times n}(\{0,1\})$.

We will denote the square tabloid obtained by applying Algorithm 4.2 to a matrix M by $\Phi(M)$. Note that $\Phi(M)$ is also defined on some matrices that do not belong to $\mathfrak{M}^{(N)}$, but in that case $\Phi(M) \notin \mathcal{T}^{(N)}$.

Example 4.4. Applying Algorithm 4.2 to the matrix below we obtain a square tabloid that is not in $\mathcal{T}^{(4)}$.

$$\Phi\left(\left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&1&1\\ 0&0&0&0&1&1&0&0\\ 0&1&1&1&0&0&0&0\\ 1&1&0&0&0&0&0&0 \end{array}\right)\right) = \begin{array}{|c|c|c|c|}\hline \chi_4&\chi_3&\chi_2&\chi_1\\ \hline \delta_4&\chi_4&\chi_2&\chi_1\\ \hline \delta_2&\delta_3&\bullet&\bullet\\ \hline \delta_1&\delta_2&\bullet&\bullet\\ \hline \end{array}$$

■

4.4 Characteristics of matrices in $\mathfrak{M}^{(N)}$ In the previous section we presented a bijective mapping from $\mathfrak{M}^{(N)}$ to $\mathcal{T}^{(N)}$. This mapping being defined by an algorithm, we can explicitely calculate its image given an element of $\mathfrak{M}^{(N)}$. However, the latter is only defined implicitely as the image of the mapping induced by the Algorithm 4.2. This section is therefore devoted to providing explicit conditions on a matrix from $\mathcal{M}_{N\times(N+m)}$ to be an element of $\mathfrak{M}_m^{(N)}$.

We will use an equivalent of Green's theorem that gives us a way of calculating the shape of the Young tableau obtained by the Robinson–Schensted correspondence from a word on the corresponding alphabet. As Robinson–Schensted correspondence is the base of Knuth's bijection, this theorem can be reformulated in terms of $\{0,1\}$-matrices.

First of all let us introduce a few notations. Let M be a $\{0,1\}$-matrix as considered above. We shall denote by $R(M,k)$ the largest possible number of 1's in M that can be arranged in k disjoint (possibly empty) sequences going North-east. Here, as in [4], we will begin each word indicating a direction with a capital letter if the sequence goes strictly in that direction, and with a small one if it does so weakly. Here, for example, "*N*orth-*e*ast" stands for "strictly North and weakly East", i.e. if two 1's in positions (i_1, j_1) and (i_2, j_2) (where $i_2 \le i_1$) belong to the same sequence then we have $i_2 < i_1$ (strictly North), and $j_1 \le j_2$ (weakly East) (see Figure 4). By convention $R(M,0) = 0$.

$$\begin{pmatrix} 1&0&\boxed{1}\\ 1&\boxed{1}&0\\ 0&\boxed{1}&0 \end{pmatrix} \qquad \begin{pmatrix} 1&0&\boxed{1}\\ \boxed{1}&\boxed{1}&0\\ 0&1&0 \end{pmatrix}$$

a b

Figure 4: The boxed sequence of 1's goes North-east on (a), but not on (b)

Taking $\lambda = (\lambda_1, \ldots, \lambda_N)$ to be the shape of the tableau P obtained from M by Knuth correspondence, we can state the following theorem:

Theorem 4.1. (Green; [5]) *In the above notations, one has:*

$$\forall k = 1 \ldots N, \quad R(M,k) - R(M,k-1) = \lambda_k.$$

Now, for $M \in \mathfrak{M}_m^{(N)}$, and denoting by $(\lambda_1, \ldots, \lambda_N)$ and $(r_1, \ldots, r_N)$ the shapes of t_λ, and r correspondingly, where $\Phi(M) = (t_\lambda, t_\mu, r)$, and taking M' to be the left-hand $N \times N$ square part of M, we obtain automatically

$$\forall k = 1 \ldots N,\quad \left\{\begin{array}{lcccl} R(M',k) &-& R(M',k-1) &=& \lambda_k\\ R(M,k) &-& R(M,k-1) &=& \lambda_k + r_k \end{array}\right. . \tag{4.17}$$

In other words, (4.17) provides us a way of calculating the shapes of t_λ and r given a $\{0,1\}$-matrix M. This immediatly delivers the first condition to be satisfied in order for M to be in $\mathfrak{M}^{(N)}$.

Condition 4.1. *Let $\Phi(M) = (t_\lambda, t_\mu, r)$, where $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $(r_1, \ldots, r_N)$ are the respective shapes of t_λ and r provided by (4.17), then for r to be a correct ribbon as described by the Definition 4.1, it is necessary that*

- *there exists $h \in [0,N]$ such that $r_k > 0$ for any $k \in [1,h]$, and $r_k = 0$ when $k > h$;*
- *$\lambda_k + r_k = \lambda_{k-1} + 1$ for all $k \in [2,h]$, and $\lambda_1 + r_1 = N$.*

The above condition, when fulfilled, guarantees that the shape of the ribbon is correct. It remains therefore to ensure that its numbering is the required one, i.e. all boxes forming the ribbon must be numbered from bottom to top, and from left to right by the sequence $\{N+1, \ldots, N+m\}$.

First of all, there has to be exactly one box in tableau P for each number between $N+1$ and $N+m$. This is obviously guaranteed by the following condition.

Condition 4.2. *For any $k \in [N+1, N+m]$ there is exactly one 1 in the k-th column of M.*

Example 4.5. Let us consider the following matrix

$$M = \left(\begin{array}{cccc|ccccc} 0&0&0&0&0&0&0&\boxed{1}&1\\ \boxed{1}&0&0&0&1&0&1&0&0\\ \boxed{1}&1&0&0&0&1&0&0&0\\ \boxed{1}&0&1&0&0&0&0&0&0 \end{array}\right)$$

Considering separately the whole matrix M and its left-hand side M', we obtain the following values:

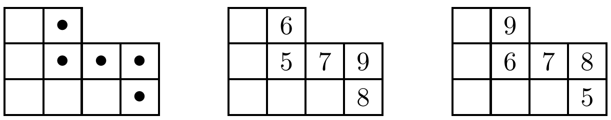
$R(M',1) = 3, \quad R(M,1) = 4$ (boxed sequence),

$R(M',2) = 4, \quad R(M,2) = 8$ (boxed and circled),

$R(M',3) = 5, \quad R(M,3) = 10$ (boxed, circled, and unmarked).

We conclude that $\lambda = (3,1,1)$, $\lambda_1 + r_1 = 4$, $\lambda_2 + r_2 = 4$, and $\lambda_3 + r_3 = 2$, i.e. $r = (1,3,1)$. It is easy to verify now that both Condition 4.1 and Condition 4.2 are satisfied.

We can deduce that the lower left-hand side tableau and the ribbon in the image of M will have the shapes shown in Figure 5-*a*. ■



a b c

Figure 5: Young diagram with ribbon (*a*); its classical (*b*) and one of the possible arbitrary numberings (*c*).

In the following we will require some additional notions. As we have seen above, given a matrix $M \in \mathcal{M}_{N\times(N+m)}$ and provided that Condition 4.1 is satisfied, we can calculate the shape of all elements of $\Phi(M)$. Thus, in particular, we know the desired classical numbering of the ribbon. For instance, the classical numbering of the ribbon in the Example 4.5 is shown in Figure 5-*b*.

Definition 4.2. *We will refer as* columns *of the ribbon to the sequences of numbers in each column of the ribbon in its classical numbering ((5,6), (7), and (8,9) in the example above).*

Definition 4.3. *Two numbers $i, j \in [N+1, N+m]$ are said to be in the* same level l*, if each one of them is exactly l boxes down from the top of its column in the classical numbering of the ribbon. We will refer as* levels *to maximal sets of numbers being in the same level. We say that level l_1 is* higher *than level l_2 if $l_1 < l_2$ — in other words if level l_1 is closer to the top.*

Example 4.6. Consider the ribbon numbered as in Figure 5-*c*, we shall say that its *columns in the classical numbering* are (5,6), (7), and (8,9) (cf. Fig. 5-*b*); its *columns in the actual numbering* are (6,9), (7), and (5,8); and its *levels [in the classical numbering]* are (6,7,9) and (5,8) (cf. Fig. 5-*b*). ■

Condition 4.3. *(ensuring that actual numbering of the ribbon is the same as the classical one)*

1. *The 1's corresponding to each column of the ribbon form a sequence going south-East.*
2. *The 1's corresponding to each level of the ribbon form a sequence going North-East.*

Indeed, Condition 4.1 allows us to obtain the exact shape of the ribbon, and therefore its classical numbering. Thus, its columns and levels are well defined, and by Condition 4.2 we can identify the 1's corresponding to each box of the ribbon in M.

Theorem 4.2. (Characterisation of $\mathfrak{M}_m^{(N)}$) *Let $M \in \mathcal{M}_{N\times(N+m)}$ be a $\{0,1\}$-matrix. $M \in \mathfrak{M}_m^{(N)}$ if and only if M satisfies all three conditions 4.1–4.3.*

4.4.1 Proof of the main theorem It is obvious that $M \in \mathcal{M}_{N\times(N+m)}$ satisfies both Conditions 4.1 and 4.2 if and only if there is exactly one box in $\Phi(M)$ numbered with each one of $\{N+1, \dots, N+m\}$ and these boxes form a correct ribbon in the sense of Definition 4.1. Thus, we only have to show that, when these two conditions are fulfilled, Condition 4.3 is equivalent to the ribbon in $\Phi(M)$ being numbered correctly.

Recall that Algorithm 4.2 is based on Robinson-Schensted-Knuth correspondence, which has column bumping as its building block. Therefore, when applying this algorithm to M, we perform a certain number Θ of column bumpings. Thus, for each $\theta \in [0, \Theta]$, one can consider a Young tableau T_θ obtained after bumping in θ boxes.

Definition 4.4. *For $a \in [N+1, N+m]$, we shall denote by $d_\theta(a)$ the column of T_θ containing the box numbered a (there is only one due to Condition 4.2). We take $d_\theta(a) = 0$ if a is not in T_θ.*

Lemma 4.1. *Suppose that Conditions 4.1 and 4.2 are satisfied, and let r be the ribbon of $\Phi(M)$. Then for any $a \geq N+1$, such that a and $a+1$ belong to the same column of r in its classical numbering, the relation $d_\theta(a) \geq d_\theta(a+1)$ is invariant over $\theta \in [0, \Theta]$ such that $d_\theta(a) > 0$.*

Proof – To prove this lemma it is sufficient to consider the two situations where the required relation (resp. the opposite one) would break, and to observe that that would require a second copy of $a+1$ (resp. a), which contradicts Condition 4.2. ■

Note 4.2. *It can be easily observed that the statement of Condition 4.3.1 is equivalent to saying that, for any a as in Lemma 4.1, a is bumped in earlier than $a+1$, i.e. $d_{\theta_a}(a) > d_{\theta_a}(a+1) = 0$, where $\theta_a = \min\{\theta | d_\theta(a) > 0\}$.*

Corollary 4.1. *In the conditions of Lemma 4.1, Condition 4.3.1 implies that* $d_\theta(a) \geq d_\theta(a+1)$ *for all* $\theta \in [0, \Theta]$.

Corollary 4.2. *Any matrix* M *in* $\mathfrak{M}^{(N)}$ *satisfies the condition 4.3.1.*

Proof – Obviously, if $M \in \mathfrak{M}^{(N)}$, we have $d_\Theta(a) = d_\Theta(a+1)$ for any a as in Lemma 4.1. As Conditions 4.1 and 4.2 also hold in this case, we can deduce from Lemma 4.1 and Note 4.2 that Condition 4.3.1 is verified. ■

We have shown therefore that Condition 4.3.1 is necessary for M to belong to $\mathfrak{M}^{(N)}$. The next step is to prove that Condition 4.3.2 is also necessary. We can remark the following by analogy with the note 4.2.

Note 4.3. *Condition 4.3.2 is equivalent to saying that, for any* a *and* b ($a < b$) *belonging to the same level in the ribbon,* b *is bumped in earlier than* a*, i.e.* $0 = d_{\theta_b}(a) < d_{\theta_b}(b)$*, where* $\theta_b = \min\{\theta | d_\theta(b) > 0\}$.

Lemma 4.2. *Suppose that Conditions 4.1, 4.2, and 4.3.1 are satisfied. If at step* $\theta+1$ *for any* a *and* b *as in Note 4.3 we have* $d_{\theta+1}(a) < d_{\theta+1}(b)$*, then the same is correct at step* θ.

Proof – The idea behind the proof is essentially the same as for Lemma 4.1: we suppose that we have the required inequality for all levels higher than a given one and obtain it for the latter by contradiction. ■

Corollary 4.3. *Any matrix* M *in* $\mathfrak{M}^{(N)}$ *satisfies the condition 4.3.2.*

Proof – As we have seen before, $M \in \mathfrak{M}^{(N)}$ implies both Conditions 4.1 and 4.2, and by Corollary 4.2 also the Condition 4.3.1. It can also be easily observed that the assumption of Lemma 4.2 holds for $\theta + 1 = \Theta$, and therefore applying this lemma recurrently, we prove the validity of Condition 4.3.2 (cf. Note 4.3). ■

Combining corollaries 4.2 and 4.3, we obtain the following proposition.

Proposition 4.2. $M \in \mathfrak{M}^{(N)}$ *implies all the conditions 4.1–4.3.*

Proposition 4.3. *If* $M \in \mathcal{M}_{N\times(N+m)}$ *satisfies all three conditions 4.1–4.3, then* $M \in \mathfrak{M}_m^{(N)}$.

Proof – As above, we shall only present a sketch of the proof in order to avoid going into technical details. First of all, observe that Conditions 4.1 and 4.2 guarantee that the shape of the ribbon is correct and its numbering is a permutation of the classical one, which leaves us to show that it is indeed the classical one.

The rest of the proof is based on the notion of plactical equivalence, and we refer the reader to [9], [10], and [12] for detailed information regarding it. We shall only mention here that if by analogy with Section 4.4 we define $R(w,k)$ — where w is a word over a totally ordered alphabet A, and $k \in \mathbb{N}$ — as a maximum total length of k increasing subsequences of w, and if we have $u \equiv v$ with '$\equiv$' denoting the plactical equivalence, then for all $k \geq 0$ we have $R(u,k) = R(v,k)$.

Let M'' be the right-hand part of M, then denoting by $w(M'')$ the column indices of 1's in M'' read in the natural (left-to-right and top-to-bottom) order, and by $w(r)$ the contents of the ribbon of $\Phi(M)$ also read in the natural order, then $\overline{w(M'')} \equiv w(r)$, where $\overline{w(M'')}$ is the mirror image of $w(M'')$. Thus we deduce that for all $k \geq 0$ we have $R(\overline{w(M'')}, k) = R(w(r), k)$.

The proof is completed now by observing that in $\overline{w(M'')}$ the maximum increasing subsequences are realised by the 1's corresponding to levels of r, and that due to the corollary 4.1 any increasing subsequence in $w(r)$ can contain at most one element of each column of r in its classical numbering. It can therefore be shown recursively — starting with shortest columns of the ribbon — that the actual numbering of r coincides with the classical one, which proves the proposition, and consequently Theorem 4.2. ■

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